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LETTER TO THE EDITOR

# An addition formula for Chebyshev polynomials

M S Milgram

Reactor Physics Branch, Chalk River Nuclear Laboratories, Chalk River, Ontario K0J 1J0, Canada

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**Abstract.** An addition formula for Chebyshev polynomials is derived, and applications suggested.

Addition formulae for classical polynomials play an important role in mathematical physics (Askey 1975a and references therein). Many results are known for both classical and non-classical polynomials (Carlson 1971, Durand 1979, Durand *et al* 1976, Koornwinder 1977, Laussen 1981, Shebalin 1979). Surprisingly, an explicit addition formula for the Chebyshev polynomials does not seem to exist in the literature, although the derivation is straightforward, if somewhat tedious. This letter is an attempt to rectify that situation.

Define

$$\mu_0 \equiv \cos\theta_0 = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi \tag{1}$$

and the Chebyshev polynomials in terms of Gegenbauer polynomials (Luke 1969),

$$T_n(\mu_0) = \lim_{\alpha\to 0}\frac{\Gamma(n+1)}{(2\alpha)_n}C_n^\alpha(\mu_0) \tag{2}$$

or trigonometric functions

$$T_n(\mu_0) = \cos n\,\theta_0. \tag{3}$$

The addition formula for $C_n^\alpha(\mu_0)$ is well known (Askey 1975a, Carlson 1971):

$$C_n^\alpha(\cos\theta_0) = \sum_{m=0}^{n}\frac{(\alpha)_m(n-m)!}{(\alpha-\frac{1}{2})_m(2\alpha+2m)_{n-m}}\sin^m\theta\sin^m\theta' \times C_{n-m}^{\alpha+m}(\cos\theta)C_{n-m}^{\alpha+m}(\cos\theta')C_m^{\alpha-1/2}(\cos\phi). \tag{4}$$

The connection between Gegenbauer polynomials of different order is also well known (Askey 1975b):

$$C_n^\lambda(x) = \sum_{k=0}^{[n/2]} a_{k,n}^\lambda C_{n-2k}^\mu(x)$$

where

$$a_{k,n}^\lambda = \frac{\Gamma(\mu)\Gamma(k+\lambda-\mu)\Gamma(n-k+\lambda)(n-2k+\mu)}{\Gamma(\lambda)\Gamma(\lambda-\mu)k!\,\Gamma(n-k+\mu+1)}. \tag{5}$$

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Substitute (5) into (4), first utilising $\lambda=\alpha+m$, then $\lambda=\alpha-\frac{1}{2}$ and use the limit (2) judiciously to eventually obtain

$$T_n(\mu_0)=\sum_{m=0}^{n}\sum_{k_1=0}^{[(n-m)/2]}\sum_{k_2=0}^{[(n-m)/2]}\sum_{k_3=0}^{[m/2]} t^{n,m}_{k_1,k_2,k_3}\sin^m\theta\sin^m\theta'$$

$$\times T_{n-m-2k_1}(\mu)T_{n-m-2k_2}(\mu')T_{m-2k_3}(\cos\phi) \tag{6}$$

where $\mu=\cos\theta$, $\mu'=\cos\theta'$:

$$t^{n,m}_{k_1k_2k_3}=\begin{cases}1 & \text{if } n=0\\[4pt] \dfrac{n}{\pi}\dfrac{4^m(n-m)!(\frac{1}{2}-m)}{\Gamma(n+m)\Gamma(m)\Gamma(m)}\dfrac{\Gamma(k_1+m)\Gamma(n-k_1)}{k_1!\Gamma(n-m-k_1+1)} & \\[4pt] \quad\times\dfrac{\Gamma(k_2+m)\Gamma(n-k_2)}{k_2!\Gamma(n-m-k_2+1)}\dfrac{\Gamma(k_3-\frac{1}{2})\Gamma(m-k_3-\frac{1}{2})}{k_3!\Gamma(m-k_3+1)} & \\[4pt] \quad\times(1-\frac{1}{2}\delta_{2k_1,n-m})(1-\frac{1}{2}\delta_{2k_2,n-m})(1-\frac{1}{2}\delta_{2k_3,m}) & \text{if } n>0.\end{cases} \tag{7}$$

Equation (1) represents the limiting case ($n=1$) of (6) which also reduces to the identity $T_n(\mu)=T_n(\mu)$ when $\theta'=0$; presumably (6) can be derived from (1) and (3) using trigonometric identities, but the calculation is not straightforward for general $n$.

Applications of this result can be found in several fields. Two notable examples are electromagnetic (Morse and Feshbach 1953) and scattering (Henry 1980) theory, where it now becomes possible to recast the usual equations resulting from expansions in spherical harmonics, into another polynomial (i.e. Chebyshev) basis with theoretically better approximation and convergence properties. The application to particle scattering will be discussed elsewhere (Milgram 1989).

## References


Askey R 1975a *Orthogonal Polynomials and Special Functions* (Philadelphia, PA: SIAM) ch 4

—— 1975b *Orthogonal Polynomials and Special Functions* (Philadelphia, PA: SIAM) p 59

Carlson B 1971 *SIAM J. Math. Anal.* **2** 347

Durand L 1979 *SIAM J. Math. Anal.* **10** 425

Durand L, Fishbane P and Simmons L 1976 *J. Math. Phys.* **17** 1933

Henry A 1980 *Nuclear Reactor Analysis* (Cambridge, MA: MIT Press) ch 8

Koornwinder T 1977 *J. Math. Anal. Appl.* **61** 136

Laussen M L 1981 *J. Phys. A: Math. Gen.* **14** 1065

Luke Y 1969 *The Special Functions and their Approximation* vol I (New York: Academic)

Milgram M 1989 The Method of Ultraspherical Harmonics in Particle Transport *Preprint* (to be presented at the ANS Topical Meeting on Advances in Nuclear Engineering Computation and Radiation Shielding, Santa Fe, April 1989)

Morse P and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill) ch 10

Shebalin J 1979 *J. Math. Phys.* **20** 1837